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A Bohr Phenomenon on the Punctured Unit Disk

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Abstract. This paper establishes the Bohr inequality with respect to the spherical chordal distance for the class of analytic functions mapping the unit disk in the complex plane into the punctured unit disk.

INTRODUCTION

Let

$$f(z)=\sum_{n=0}^{\infty}a_n z^n$$

be analytic in the unit disk $U:=\{z:|z|<1\}$ of the complex plane satisfying $|f(z)|<1$ in U. Bohr [1] in 1914 showed that these functions satisfy

$$\sum_{n=0}^{\infty}|a_n z^n|\leq 1 \tag{1}$$

in the disk $|z|\leq 1/6$. This result became known as the Bohr theorem. The radius $1/6$ was later improved independently to the sharp constant $1/3$ by Wiener, Riesz and Schur (see [2-4]). Analogous results to the Bohr for several complex variables have been established by replacing U with a complete Reinhardt domain [5], a unit ball or hypercone in higher dimensions [6].

An equivalent form for (1) is

$$\sum_{n=1}^{\infty}|a_n z^n|\leq 1-|a_0|=d(a_0,\partial U),$$

where d and ∂U denote respectively the Euclidean distance and the boundary of U. Generally, a class of analytic (or harmonic) functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ in U is said to have a Bohr phenomenon (first introduced in [7]) if there exists a number $0<\rho_0\leq 1$ satisfying

$$\sum_{n=1}^{\infty}|a_n z^n|\leq d(a_0,\partial f(U))$$

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for all $|z|<\rho_0$. The constant ρ_0 is then known as the Bohr radius for this class of functions (with respect to the Euclidean distance). Under this definition, the Bohr radius $1/3$ has been found to hold for the class of convex functions [8] and the class of analytic functions mapping U into the exterior of the closed unit disk [9].

This paper discusses the Bohr phenomenon for the class H_0 consisting of analytic functions mapping U into the punctured unit disk $U_0 = U \setminus \{0\}$. Instead of the Euclidean distance, the following spherical chordal distance will be used:

$$\lambda(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

Note that $\lambda(a_0, \partial U_0) = \min\{\lambda(|a_0|, 0), \lambda(|a_0|, 1)\}$ and

$$\frac{\lambda(|a_0|, 0)}{\lambda(|a_0|, 1)} = \frac{\sqrt{2}|a_0|}{1-|a_0|}\begin{cases} >1, & \text{if } |a_0| > \sqrt{2}-1, \\ <1, & \text{if } |a_0| < \sqrt{2}-1, \\ =1, & \text{if } |a_0| = \sqrt{2}-1. \end{cases}$$

Thus

$$\lambda(a_0, \partial U_0) = \begin{cases} \lambda(|a_0|, 1), & \text{if } |a_0| > \sqrt{2}-1, \\ \lambda(|a_0|, 0), & \text{if } |a_0| < \sqrt{2}-1, \\ \lambda(|a_0|, 0) = \lambda(|a_0|, 1), & \text{if } |a_0| = \sqrt{2}-1. \end{cases}$$

Given an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, define

$$f^*(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

It is evident that both power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} |a_n| z^n$ have the same radius of convergence. The following properties will be needed in the sequel. Let $g_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions. Then

$$\begin{aligned}(g_1 + g_2)^*(|z|) &= \sum_{n=0}^{\infty} |a_n + b_n||z|^n \le \sum_{n=0}^{\infty} (|a_n| + |b_n|)|z|^n \\ &= \sum_{n=0}^{\infty} |a_n||z|^n + \sum_{n=0}^{\infty} |b_n||z|^n = g_1^*(|z|) + g_2^*(|z|),\end{aligned}$$

and

$$\begin{aligned}(g_1 g_2)^*(|z|) &= \sum_{n=0}^{\infty} \left|\sum_{k=0}^{n} a_{n-k} b_k\right| |z|^n \le \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} |a_{n-k}||b_k|\right)|z|^n \\ &= \left(\sum_{n=0}^{\infty} |a_n||z|^n\right)\left(\sum_{n=0}^{\infty} |b_n||z|^n\right) = g_1^*(|z|) g_2^*(|z|).\end{aligned}$$

MAIN RESULTS

LEMMA 1. Let f be an analytic function mapping the unit disk U into itself. If $\sqrt{2}-1 \le |f(0)| < 1$, then $f^*(z) \in U_0$ for $|z| < 1/3$.

PROOF. Write $f(z) = a + \sum_{n=1}^{\infty} a_n z^n$ and without loss of generality, assume that $\sqrt{2}-1 \le a < 1$. Let $\omega = \exp\left(\frac{2\pi i}{n}\right)$ be the n-th root of unity. As U is convex and $\sum_{k=1}^{n} \omega^k = 0$, the analytic function

$$\begin{aligned}\varphi(z^n) &= \frac{1}{n}\sum_{k=1}^{n} f(\omega^k z) = \frac{1}{n}\left[f(\omega z) + f(\omega^2 z) + \cdots + f(\omega^n z)\right] \\ &= \frac{1}{n}\left[na + \left(\sum_{k=1}^{n}\omega^k\right)a_1 z + \left(\sum_{k=1}^{n}\omega^k\right)a_2 z^2 + \cdots + na_n z^n + \cdots\right] \\ &= a + a_n z^n + a_{2n} z^{2n} + \cdots = a + \sum_{m=1}^{\infty} a_{mn} z^{mn}\end{aligned}$$

has its image in U. Letting $z^n = \zeta$, the function

$$\varphi(\zeta) = a + \sum_{m=1}^{\infty} a_{mn}\zeta^m$$

is again an analytic function mapping U into itself. Applying the Schwarz-Pick Lemma gives

$$|a_n| = \varphi'(0) \le 1 - |\varphi(0)|^2 = 1 - a^2.$$

Thus for $|z| < 1/3$,

$$|f^*(z)| \ge a - \sum_{n=1}^{\infty}|a_n||z|^n > a - \frac{1-a^2}{2} = \frac{2a-1+a^2}{2}, \tag{2}$$

and

$$|f^*(z)| \le a + \sum_{n=1}^{\infty}|a_n||z|^n < a + \frac{1-a^2}{2} = \frac{2a+1-a^2}{2}. \tag{3}$$

Since $a \ge \sqrt{2}-1$ implies $2a-1+a^2/2 \ge 0$, inequality (2) gives $|f^*(z)| > 0$. On the other hand, $a < 1$ implies $2a+1-a^2/2 < 1$ and so inequality (3) gives $|f^*(z)| < 1$. ∎

REMARK 2. The inequalities (2) and (3) yield the following important inequality:

$$\frac{2a-1+a^2}{2} < f^*(|z|) = a + \sum_{n=1}^{\infty}|a_n||z|^n < a + \frac{1-a^2}{2} = \frac{2a+1-a^2}{2}. \tag{4}$$

REMARK 3. The condition in LEMMA 1 cannot be removed. Consider the Möbius transformation φ in U,

$$\varphi(z)=\frac{z+a}{1+az}=a+(1-a^2)\sum_{n=1}^{\infty}(-a)^{n-1}z^n, \qquad 0<a<\left(\sqrt{17}-3\right)/4<1/\sqrt{2}.$$

Then

$$\varphi^*(z)=\frac{z+a}{1+az}=a+(1-a^2)\sum_{n=1}^{\infty}a^{n-1}z^n=2a+\frac{z-a}{1-az}=\frac{a+\left(1-2a^2\right)z}{1-az}.$$

Note that

$$\varphi^*(z)=\frac{a+\left(1-2a^2\right)z}{1-az}=0$$

if and only if $a+\left(1-2a^2\right)z=0$, that is, for $z=-a/\left(1-2a^2\right)$. However, $a/\left(1-2a^2\right)<1/3$ holds for $0<a<\left(\sqrt{17}-3\right)/4\approx 0.28077$ which implies $\varphi^*(z_0)=0$ for some $|z_0|<1/3$.

LEMMA 4. If $f\in H_0$, then for $|z|<1/3$,

$$\frac{\lambda\left(|f(0)|,f^*(|z|)\right)}{\lambda\left(f(0),\partial U_0\right)}\begin{cases}>1, & 0<|f(0)|<\sqrt{2}-1,\\ <1, & \sqrt{2}-1\le|f(0)|<1.\end{cases}$$

PROOF. Write $f(z)=a+\sum_{n=1}^{\infty}a_nz^n$ where $|z|<1/3$. Suppose that $0<a<\sqrt{2}-1$. Then

$$\lambda(a,\partial U_0)=\lambda(a,0)=\frac{a}{\sqrt{\left(1+a^2\right)}}.$$

Note that

$$\lambda\left(a,\left(2a-1+a^2\right)/2\right)=\frac{1-a^2}{\sqrt{\left(1+a^2\right)\left(4+\left(2a-1+a^2\right)^2\right)}}.$$

Hence

$$\frac{\lambda\left(a,\left(2a-1+a^2\right)/2\right)}{\lambda(a,\partial U_0)}=\frac{1/a-a}{\sqrt{4+\left(2a-1+a^2\right)^2}}>1 \tag{5}$$

provided $1/a^2+a^2-6-\left(2a-1+a^2\right)^2>0$, which is true for all $a\in\left(0,\sqrt{2}-1\right)$. Thus (4) and (5) give

$$\lambda\left(a,f^*(|z|)\right)>\lambda\left(a,\left(2a-1+a^2\right)/2\right)>\lambda(a,\partial U_0).$$

Suppose now that $\sqrt{2}-1\le a<1$ Then

$$\lambda(a,\partial U_0)=\lambda(a,1)=\frac{1-a}{\sqrt{2\left(1+a^2\right)}}.$$

Note that

$$\lambda\left(a,\left(2a+1-a^2\right)/2\right)=\frac{1-a^2}{\sqrt{\left(1+a^2\right)\left(4+\left(2a+1-a^2\right)^2\right)}}.$$

Hence

$$\frac{\lambda\left(a,\left(2a+1-a^2\right)/2\right)}{\lambda\left(a,\partial U_0\right)}=\frac{\sqrt{2}(1+a)}{\sqrt{4+\left(2a+1-a^2\right)^2}}<1 \tag{6}$$

provided $a^4-4a^3+3=(1-a)\left(3+3a+3a^2-a^3\right)>0$. As $3+3a+3a^2-a^3>0$ for $a\in\left[\sqrt{2}-1,1\right)$, it follows that $a^4-4a^3+3>0$ for all $a\in\left[\sqrt{2}-1,1\right)$. Thus (4) and (6) give

$$\lambda\left(a,f^*(|z|)\right)<\lambda\left(a,\left(2a+1-a^2\right)/2\right)<\lambda\left(a,\partial U_0\right). \qquad \blacksquare$$

Define two subclasses of H_0 as follows:

$$H_1=\left\{f\in H_0:0<|f(0)|<\sqrt{2}-1\right\}$$

and

$$H_2=\left\{f\in H_0:\sqrt{2}-1\leq|f(0)|<1\right\},$$

where $H_1\cup H_2=H_0$ and $H_1\cap H_2=\phi$. As mentioned earlier, the Bohr phenomenon stated in the following results considers the spherical chordal distance λ instead of the usual Euclidean distance.

THEOREM 5. The class H_2 has Bohr phenomenon and the Bohr radius is $1/3$. The Bohr phenomenon does not occur in the class H_1.

PROOF. The result follows from LEMMA 4. To show the Bohr radius is $1/3$, consider the function

$$f(z)=\exp\left(-a\frac{1+z}{1-z}\right), \qquad z\in U$$

for some real a so that $\sqrt{2}-1\leq e^{-a}<1$. The Taylor series expansion for f is

$$\begin{aligned}\exp\left(-a\frac{1+z}{1-z}\right)&=\exp\left(-a-2a\sum_{n=1}^{\infty}z^n\right)=\frac{1}{e^a}\exp\left(-2a\sum_{n=1}^{\infty}z^n\right)\\&=\frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\frac{(-2a)^m}{m!}\left(\sum_{n=1}^{\infty}z^n\right)^m\\&=\frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\frac{(-2a)^m}{m!}\sum_{n=m}^{\infty}\left(\sum_{s_1+s_2+\cdots s_m=n}1\right)z^n,\end{aligned}$$

where $s_1,s_2,\ldots,s_m$ are positive integers and the latter sum is taken over all m-tuples $(s_1,s_2,\ldots,s_m)$ satisfying $s_1+s_2+\cdots+s_m=n$. Thus

$$f(z)=\frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{(-2a)^m}{m!}\binom{n-1}{m-1}z^n$$
$$=\frac{1}{e^a}+\frac{1}{e^a}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2a)^m}{m!}\binom{n-1}{m-1}\right]z^n,$$

which yields

$$f^*(|z|)=\frac{1}{e^a}-\frac{1}{e^a}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2a)^m}{m!}\binom{n-1}{m-1}\right]|z|^n=\frac{2}{e^a}-f(|z|).$$

Thus for $z\in U$,

$$d\left(e^{-a},f^*(|z|)\right)=f^*(|z|)-e^{-a}\geq e^{-a}-f(|z|)=d\left(e^{-a},f(|z|)\right),$$

and it follows that

$$\lambda\left(e^{-a},f^*(|z|)\right)\geq\lambda\left(e^{-a},f(|z|)\right) \tag{7}$$

by the definition of λ. Let $b=f(0)=e^{-a}>0$. Since $a=-\log b$, f can be written as

$$f(z)=\exp\left(\frac{1+z}{1-z}\log b\right)=b\exp\left(\frac{2z}{1-z}\log b\right)=b^{\frac{1+z}{1-z}}. \tag{8}$$

Let $|z|=r$. Note that

$$\frac{\lambda\left(b,b^{\frac{1+r}{1-r}}\right)}{\lambda(b,1)}=\frac{\sqrt{2}b}{\sqrt{1+b^{\frac{1+r}{1-r}}}}\times\frac{1-b^{\frac{2r}{1-r}}}{1-b}\to\frac{2r}{1-r}>1$$

if and only if $r>1/3$ as $b\to 1$. Consequently, together with (7) and (8),

$$\lambda\left(f^*(r),b\right)\geq\lambda\left(b,b^{\frac{1+r}{1-r}}\right)>\lambda(b,1)=\lambda(b,\partial U_0)$$

for $r>1/3$ as $b\to 1$. Thus the value $1/3$ is best possible. ■

COROLLARY 6. The Bohr phenomenon does not occur in the class H_0.
PROOF. THEOREM 5 and the fact that $H_1\subset H_0$. ■

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REFERENCES

1. H. Bohr, Proc. Lond. Math. Soc. (3) **13**, 1–5 (1914).
2. V. I. Paulsen, G. Popescu, D. Singh, Proc. Lond. Math. Soc. (3) **85**, 493–512 (2002).

3. S. Sidon, Math. Z. **26**, 731–732 (1927).
4. M. Tomić, Math. Scand. **11**, 103–106 (1962).
5. L. Aizenberg, "Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series" in *Selected Topics in Complex Analysis*, edited by V. Y. Eiderman,and M. V. Samokhin (Birkhäuser, Basel, 2005), pp. 87–94.
6. L. Aizenberg, Proc. Amer. Math. Soc. **128**, 1147–1155 (2000).
7. C. Bénéteau, A. Dahlner, D. Khavinson, Comput. Methods Funct. Theory **4**, 1–19 (2004).
8. L. Aizenberg, Stud. Math. **180**, 161–168 (2007).
9. Y. Abu Muhanna, R. M. Ali, J. Math. Anal. Appl. **379**, 512--517 (2011).